SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class:

2022

YEAR 12
TASK 4
TRIAL HSC

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks (pages 2 – 5)

Attempt Questions 1 – 10
Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 15)

Attempt all Questions in Section II
Allow about 2 hours and 45 minutes for this section

Examiner: JJ

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section.
Use the multiple-choice answer sheet for Questions 1–10.

---

1 The polynomial $Q(z)$ has real coefficients and $z = -3 + 2i$ is a zero of $Q(z)$.

Which quadratic polynomial must be a factor of $Q(z)$?

A. $z^2 + 6z + 13$

B. $z^2 + 6z + 5$

C. $z^2 - 6z + 13$

D. $z^2 - 6z + 5$

2 If $z = e^{\frac{7\pi i}{12}}$, which of the following is equivalent to $\text{Im}\left(z^3 + 1\right)$?

A. $\dfrac{\sqrt{2}}{2}$

B. $\dfrac{2+\sqrt{2}}{2}$

C. $-\dfrac{\sqrt{2}}{2}$

D. $-\dfrac{2+\sqrt{2}}{2}$

3 What is an equation of a circle on an Argand diagram that goes through the points represented by the numbers $2 - i$ and $-3 + 4i$?

A. $|z-3+i| = 5$

B. $|z-3-i| = 25$

C. $|z+3+i| = 25$

D. $|z+3+i| = 5$

4 Which of the following is equivalent to $\int \frac{2-x}{\sqrt{4-x^2}}\,dx$?

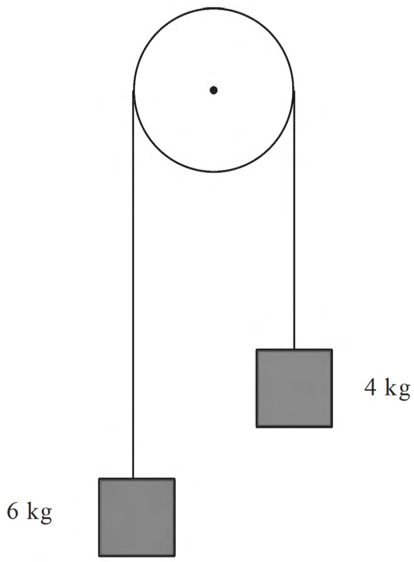
A. $\frac{\ln(4-x^2)}{4}+C$

B. $\frac{\ln(4-x^2)}{4}-4\sin^{-1}\left(\frac{x}{2}\right)+C$

C. $\sqrt{4-x^2}-\sin^{-1}\left(\frac{x}{2}\right)+C$

D. $\sqrt{4-x^2}+2\sin^{-1}\left(\frac{x}{2}\right)+C$

5 A light inextensible string passes over a smooth pulley.
Attached to each end of the strings are masses of 4 kg and 6 kg, as shown.



What is the acceleration of the larger mass?

A. $3g$ B. $\frac{3g}{10}$ C. $\frac{g}{5}$ D. $\frac{g}{10}$

6 A particle undergoes simple harmonic motion according to

$$v^2 = 5 + 10x - 4x^2.$$

What is the period of the harmonic motion?

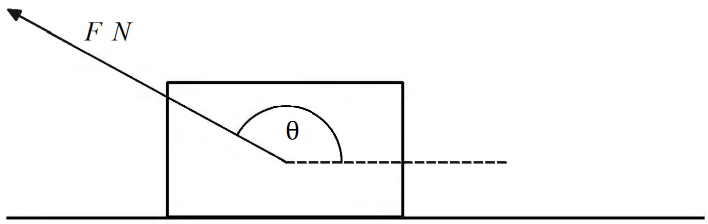
A. π

B. $\dfrac{\pi}{2}$

C. 2π

D. $\dfrac{5\pi}{2}$

7 A block of mass $2\sqrt{2}$ kg, is pulled along a smooth horizontal plane by a force F as shown in the diagram below.



Which of the following set of conditions produces the largest magnitude of acceleration for the mass?

The information is presented as (F, θ).

A. $\left(\sqrt{2}, \pi\right)$

B. $\left(2\sqrt{3}, \dfrac{5\pi}{6}\right)$

C. $\left(2\sqrt{3}, \dfrac{3\pi}{4}\right)$

D. $\left(5, \dfrac{2\pi}{3}\right)$

8 Which of the following integrals is equivalent to zero?

A. $\displaystyle\int_0^{2\pi} |\sin x| \; dx$ B. $\displaystyle\int_0^{2\pi} \sin|x|\, dx$

C. $\displaystyle\int_0^{2\pi} x\sin x \, dx$ D. $\displaystyle\int_0^{2\pi} x^2 \sin x \, dx$

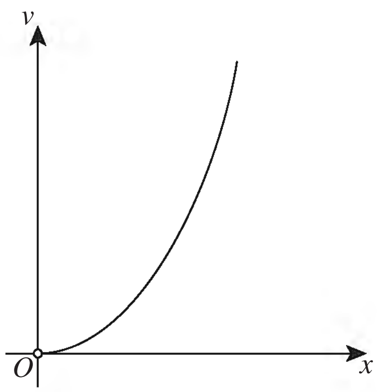
9 The complex number z satisfies $|z-4-6i|-|z+2|=0$.
What is the minimum value of $|z|$?

A. 2 B. $2\sqrt{2}$ C. 3 D. $2\sqrt{3}$

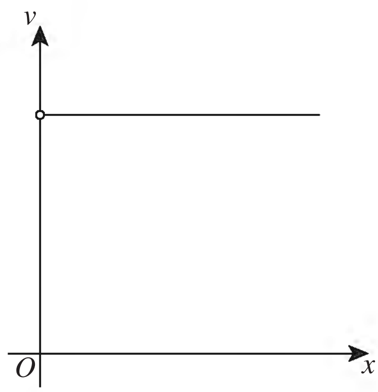
10 A particle moves in a straight line such that its acceleration a is given by $a = vx$, where v is the particle's velocity, x is the particle's position, and $v, x > 0$.

Which of the following diagrams best shows the relationship between v and x?

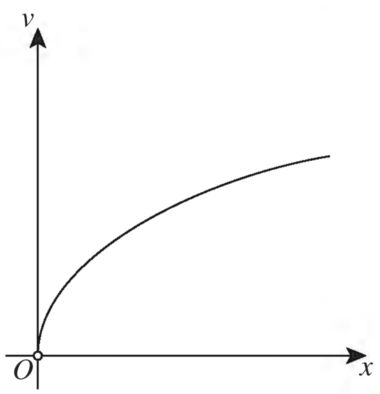
A.



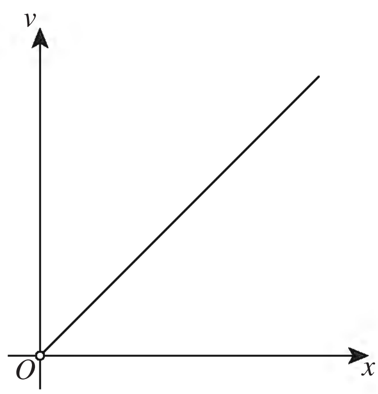
B.



C.



D.



Section II

90 marks
Attempt Questions 11-16
Allow 2 hours and 45 minutes for this section

In Questions 11-16, your responses should include ALL relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use a SEPARATE writing booklet

(a) Let z and w be complex numbers such that $z = 3 - 4i$ and $w = 4 + 5i$.
Find the following in $x + iy$ form where $x, y \in \mathbb{R}$.

(i) $4z - 3w$ **1**

(ii) $\dfrac{w}{z}$ **1**

(iii) $\overline{z}w - z\overline{w}$ **1**

(iv) $\sqrt{z}$, where $\sqrt{z} = a + ib, a > 0$. **2**

(b) Find the exact values of r and θ, where $r \in \mathbb{R}^+$, $-\pi < \theta \leq \pi$, and **2**

$$re^{i\theta} = -1 - \sqrt{5} - i\sqrt{10 - 2\sqrt{5}}.$$

(c) Given a complex number z, such that **2**

$$z = (2i + 1)t^2 - (1 + 5i)t - 3(2 + i).$$

Find the value(s) of t if z is a non-zero real number and t is a real number.

Question 11 is continued on page 7

Question 11 (continued)

(d) Sketch the region in the complex plane where **3**

$$0 < \arg(z-1-i) \le \frac{\pi}{3} \text{ and } |z-1-i| < \sqrt{2}$$

(e) Let $S = i + 2i^2 + 3i^3 + 4i^4 + \ldots + 100i^{100}$.

(i) Find and simplify $S - iS$. **2**

(ii) Hence, find and simplify S. **2**

End of Question 11

Question 12 (16 marks) Use a SEPARATE writing booklet

(a) The motion of a particle moving along a straight line is given by the equation **2**

$$\frac{d^2x}{dt^2} = -9x.$$

If $x = 0$ and $v = -9$ when $t = 0$, find its displacement at any time t in the form $x = A\sin\left(nt + \phi\right)$, where $n, A \in \mathbb{R}^+$ and $\phi \in \left[0, \pi\right]$.

(b) Find (i) $\int \sec^3 x \tan x \, dx$ **1**

(ii) $\int \frac{5}{(x+2)(x-3)} dx$ **2**

(c) Evaluate the following definite integrals:

(i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x \, dx$. **3**

(ii) $\int_{-1}^{1} \frac{\sin\frac{2\pi}{5}}{x^2 + 2x\cos\frac{2\pi}{5} + 1} dx$. **3**

(Hint: Express the denominator in the form $\left(x+a\right)^2 + b^2$)

(d) Consider the two integrals $I = \int_{-1}^{1} \frac{1}{\left(1+x^2\right)^2} dx$ and $J = -\int_{-1}^{1} \frac{t^2}{\left(1+t^2\right)^2} dt$.

(i) Explain why the use of the substitution $x = \frac{1}{t}$ does not demonstrate that $I = J$. **1**

(ii) Evaluate I. **2**

(iii) Evaluate J. **2**

End of Question 12

Question 13 (13 marks) Use a SEPARATE writing booklet

(a) A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. **3**

If $\ddot{x} = -2e^{-x}$ and $v = 2$, $x = 0$ when $t = 0$, find x as a function of t.

(b) (i) Let $D(x) = x - \tan^{-1}(x)$. **2**

By finding $D'(x)$ and using $D(0) = 0$, show that $D(x) > 0$ for all $x > 0$.

(ii) A particle of unit mass is projected from the ground level with a velocity of u at an angle of α to the horizontal. **3**

The horizontal and vertical velocity at time t seconds are $\dot{x}$ m/s and $\dot{y}$ m/s respectively.

The body experiences air resistance such that the equation of motion is given by

$$\ddot{\underset{\sim}{r}} = -k\dot{x}^2\,\underset{\sim}{i} - \left(g + k\dot{y}^2\right)\underset{\sim}{j},$$

where g is the magnitude of the acceleration due to gravity.

The particle reaches a maximum height in T_1 seconds.

Show that $T_1 = \dfrac{1}{\sqrt{kg}}\tan^{-1}\left(\sqrt{\dfrac{k}{g}}\,u\sin\alpha\right)$.

(iii) If it achieves maximum height in T_2 seconds in the absence of air resistance, show using (i), or otherwise, that $T_2 > T_1$. **2**

You may assume that the displacement of the particle is given by

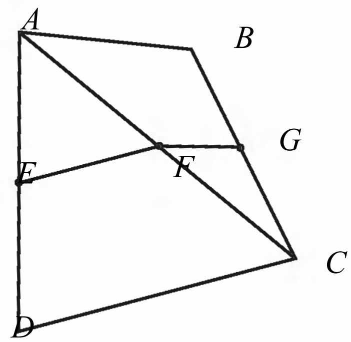
$$r = ut\cos\alpha\,\underset{\sim}{i} + \left(ut\sin\alpha - \frac{1}{2}gt^2\right)\underset{\sim}{j}$$. Do NOT prove.

Question 13 continues on page 11

Question 13 (continued)

(c) $ABCD$ is a quadrilateral. **3**

E, F and G are the midpoints of AD, AC, and BC respectively.



Using vector methods, show that $\overrightarrow{AB} + \overrightarrow{DC} = 2\,\overrightarrow{EG}$.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) Use mathematical induction to prove **3**

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}>\sqrt{n}\text{, where } n\in\mathbb{Z} \text{ and } n>1.$$

(b) Find $\displaystyle\int \frac{\cos x}{\sin x\sqrt{1+\sin x}}\,dx$. **3**

(c) A particle of mass m falls vertically from rest under gravity, where the resistance to the motion has magnitude $\dfrac{mv^2}{g}$ N when the speed of the particle is v m/s.

The acceleration due to gravity is g m/s^2.

At time t seconds the particle has fallen x metres and has velocity v m/s.

(i) Show that $\ddot{x}=\dfrac{\left(g^2-v^2\right)}{g}$. **1**

(ii) Show that $v=g\left(\dfrac{e^{2t}-1}{e^{2t}+1}\right)$ and $v^2=g^2\left(1-e^{-\frac{2x}{g}}\right)$. **5**

(iii) Find in simplest exact form the time taken and the distance fallen by the particle in reaching half of its terminal velocity. **3**

End of Question 14

Question 15 (16 marks) Use a SEPARATE writing booklet

(a) Let $t = \tan \dfrac{x}{2}$.

(i) Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+k\sin x}\,dx = \frac{2}{\sqrt{1-k^2}}\tan^{-1}\left(\sqrt{\frac{1-k}{1+k}}\right)$, where $0 < k < 1$. **4**

Let $\displaystyle I_n = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{2+\sin x}\,dx$, where $n = 0, 1, 2, \ldots$.

(ii) Show that $\displaystyle I_{n+1} + 2I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$. **1**

(iii) Using parts (i) and (ii), find the value of I_2. **3**
Give your answer in the form $m\pi + 1$, where m is irrational in simplest form.

(b) Let $P(z) = z^7 - 1$.

(i) Write your solutions to $P(z) = 0$ on an Argand diagram. **1**

(ii) Show that $P(z) = z^3(z-1)\left[\left(z+\frac{1}{z}\right)^3 + \left(z+\frac{1}{z}\right)^2 - 2\left(z+\frac{1}{z}\right) - 1\right]$ **3**

(iii) Hence, solve the equation $x^3 + x^2 - 2x - 1 = 0$ **2**

(iv) Hence prove that $\operatorname{cosec}\dfrac{\pi}{14}\operatorname{cosec}\dfrac{3\pi}{14}\operatorname{cosec}\dfrac{5\pi}{14} = 8$. **2**

End of Question 15

Question 16 (14 marks)   Use a SEPARATE writing booklet

(a) Let $S_n = 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^n \frac{1}{2n+1}$, where $n \in \mathbb{Z}^+$.

(i) State the sum of $1 - x^2 + x^4 - \ldots + (-1)^n x^{2n}$. **1**

(ii) Find a real constant A, so that $S_n = A + (-1)^n \int_0^1 \frac{x^{2n+2}}{1+x^2}\, dx$. **2**

(iii) Deduce that $\left| S_n - A \right| < \frac{1}{2n+3}$. **2**

(iv) Find $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ **1**

(b) Suppose that for $a_i > 0$ for all $i > 0$ such that $i \in \mathbb{Z}$.

(i) By considering $a_1 a_2 - \left(\frac{a_1 + a_2}{2}\right)^2$, show that $a_1 a_2 \leq \left(\frac{a_1 + a_2}{2}\right)^2$. **1**

(ii) Prove by induction that for all positive integers m **4**

$$a_1 \times a_2 \times a_3 \times \ldots \times a_{2^m - 1} \times a_{2^m} \leq \left(\frac{a_1 + a_2 + a_3 + \ldots + a_{2^m - 1} + a_{2^m}}{2^m}\right)^{2^m}.$$

(iii) If $n < 2^m$, let $b_i = a_i$ for all $i = 1, 2, \ldots, n$ and let $b_{n+1} = \ldots = b_{2^m} = A$, **1**

where $A = \frac{1}{n}\sum_{i=1}^{n} a_i$.

By applying part (ii) to the b_i, or otherwise, show that $a_1 \times \ldots \times a_n \times A^{2^m - n} \leq A^{2^m}$.

(iv) Deduce the inequality $\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$. **2**

End of paper

SYDNEY
BOYS
HIGH
SCHOOL

2022

YEAR 12
TASK 4
TRIAL HSC

Mathematics Extension 2

Sample Solutions

NOTE: Some of you may be disappointed with your mark.

This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.

Just because you have shown 'working' does not justify that your solution is worth any marks.

MC Answers

1	A	**6**	A
2	C	**7**	B
3	D	**8**	B
4	D	**9**	B
5	C	**10**	A

Multiple Choice Solutions

1 **A**

Eliminate options B and D re constant term.

$$\alpha+\beta = -3+2i+-3-2i = -6$$
$$\alpha\beta = (-3+2i)(-3-2i) = 9+4 = 13$$

$$\therefore z^2-(\alpha+\beta)z-\alpha\beta = z^2+6z+13$$

2 **C**

Note: $\text{Im}(z^3+1) = \text{Im}(z^3)+\text{Im}(1) = \text{Im}(z^3)$ then options B and D can be eliminated.

$$z^3 = e^{\frac{7\pi i}{4}} = \cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$$

$$\text{Im}(z^3+1) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

3 **D**

B & C are out due to the "25".
Test A and D on both points if necessary

Test A: with $(2 - i)$: $\left|(2-i)-3+i\right| \neq 5$

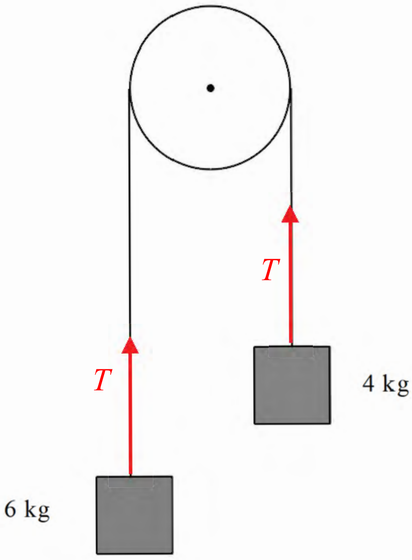
4 **D**

A and B are obviously out as there is no $\dfrac{f'(x)}{f(x)}$

$$\int \frac{2-x}{\sqrt{4-x^2}}\,dx = \int \frac{2}{\sqrt{4-x^2}}\,dx + \frac{1}{2}\int \frac{-2x}{\sqrt{4-x^2}}\,dx$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right)+\sqrt{4-x^2}+C$$

5 **C**



Let a = acceleration of the system

$$4a = T - 4g \quad -(1)$$
$$6a = 6g + T \quad -(2)$$

(1) + (2): $10a = 2g$

$$a = \frac{g}{5}$$

6 **A**

$$v^2 = 5 + 10x - 4x^2 = 4(\cdots)$$
$$\therefore n = 2$$
$$\therefore T = \frac{2\pi}{n} = \pi$$

7 **B**

The largest acceleration would be produced by the greatest force.
We need F to be as large as possible with θ closest to 180° if possible.

So C is eliminated in favour of B

Even though option A has 180° with it $\left|2\sqrt{3} \times -\frac{\sqrt{3}}{2}\right| = 3$.

So A is eliminated.

With option D we get $\left|5 \times -\frac{1}{2}\right| = 2.5$.

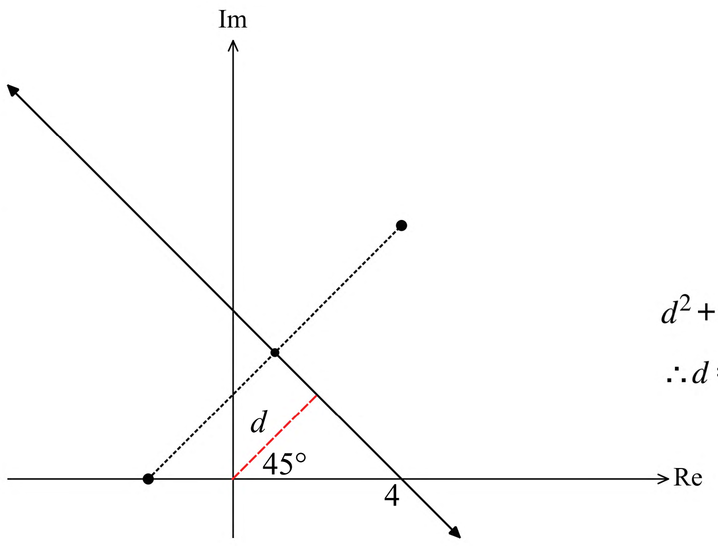
8 **B**

In B the absolute value is irrelevant.
A – the graph is always even.
C – the graph is trapped between $y = x$ and $y = -x$, so the symmetry with the x-axis isn't there.
D – odd function but the domain is wrong.

9 **B**

$|z-4-6i| = |z+2|$ (This is the perpendicular bisector of $4 + 6i$ and -2)

Midpoint is (1, 3), and so the equation is $y = -x + 4$.



$d^2 + d^2 = 16 \Rightarrow d^2 = 8$

$\therefore d = 2\sqrt{2}$

10 **A**

$$a = v\frac{dv}{dx} = vx$$

$$\therefore \frac{dv}{dx} = x$$

$$\therefore v = \frac{1}{2}x^2 + C$$

So the graph must be parabolic.

Question 11

a) i) $4z - 3w = 4(3-4i) - 3(4+5i)$
$= 12 - 16i - 12 - 15i$
$= -31i$

ii) $\frac{w}{z} = \frac{4+5i}{3-4i} \times \frac{3+4i}{3+4i}$

$= \frac{12 + 16i + 15i - 20}{9 + 16}$

$= -\frac{8}{25} + \frac{31}{25}i$

iii) $\bar{z}w - z\bar{w} = (\overline{3-4i})(4+5i) - (3-4i)(\overline{4+5i})$

$= (3+4i)(4+5i) - (3-4i)(4-5i)$
$= \cancel{12}+15i+16i\cancel{-20} - (\cancel{12}-15i-16i\cancel{-20})$
$= 62i$

iv) $\sqrt{z} = a+ib$ where $a, b \in \mathbb{R}$ & $a > 0$
$z = a^2 + 2abi - b^2$
$3-4i = a^2-b^2 + 2abi$
equate
$a^2-b^2 = 3$ —— ①
$2ab = -4$ —— ②
$(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$
$= 3^2 + (-4)^2$
$= 25$
$\therefore a^2+b^2 = 5$ —— ③ since $a, b \in \mathbb{R}$
① + ③
$2a^2 = 8$
$a^2 = 4$

$a = +2$ since $a > 0$

sub into ②

$2(2)b = -4$

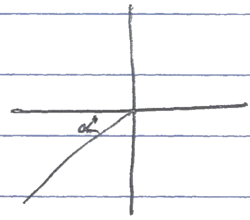
$b = -1$

$\therefore \sqrt{z} = 2 - i$

COMMENT: This question was mostly done well. Some students wrote two answers for (iv). They were not penalised for this.

b) $-1-\sqrt{5} - i\sqrt{10-2\sqrt{5}}$

lies in the 3rd quadrant.

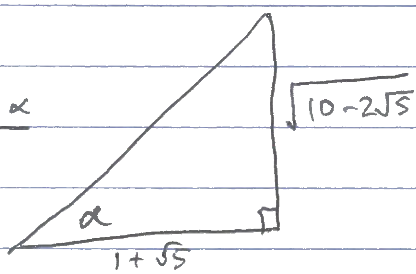


$r = \sqrt{(-1-\sqrt{5})^2 + (-\sqrt{10-2\sqrt{5}})^2}$

$= \sqrt{1 + \cancel{2\sqrt{5}} + 5 + 10 - \cancel{2\sqrt{5}}}$

$= 4$

Acute angle α



$\tan\alpha = \dfrac{\sqrt{10-2\sqrt{5}}}{1+\sqrt{5}}$

$\alpha = 36°$

$\alpha = \dfrac{\pi}{5}$

$\theta = -\pi + \dfrac{\pi}{5}$

$= -\dfrac{4\pi}{5}$

COMMENT: Most students could find r.
Most students could not find θ as they did not get the right quadrant.

c) $z = (2i+1)t^2 - (1+5i)t - 3(2+i)$

$$= (t^2 - t - 6) + i(2t^2 - 5t - 3)$$

$$= (t-3)(t+2) + i(2t+1)(t-3)$$

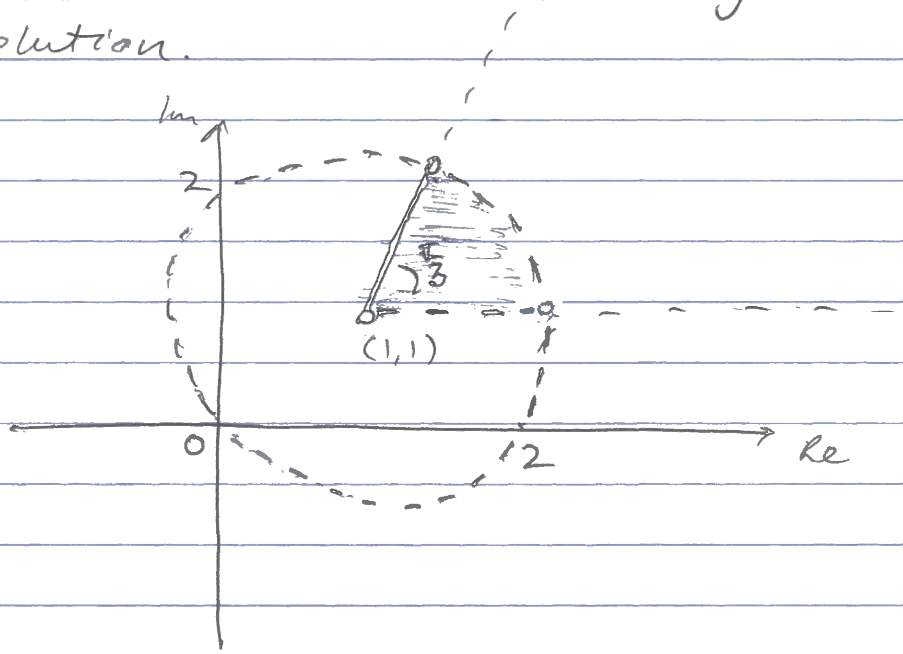
z is a non-zero real number

ie $\operatorname{Im}(z) = 0$ and $\operatorname{Re}(z) \neq 0$

$\therefore t = -\frac{1}{2}$ only.

COMMENT: This was not done particularly well. Students lost half a mark if they included $t = 3$ as a solution.

d)



COMMENT:
All information should be indicated on diagram.
Circle: centre $(1,1)$ radius $\sqrt{2}$
Arguments from $(1,1)$ between 0 and $\frac{\pi}{3}$.

e) i) $S = i + 2i^2 + 3i^3 + 4i^4 + \dots + 100i^{100}$ ——— ①

$iS = i^2 + 2i^3 + 3i^4 + 4i^5 + \dots + 100i^{101}$ ——— ②

① − ②

$S - iS = i + i^2 + i^3 + i^4 + \dots + i^{100} - 100i^{101}$

$= (i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + \dots + (i^{97} + i^{98} + i^{99} + i^{100}) - 100i^{101}$

$= 0 + 0 + \dots + 0 + -100(i^4)^{25} i$

$= -100(1)^{25} i$

$= -100i$

Note: $i + i^2 + i^3 + i^4 = i + (-1) + (-i) + 1$
$= 0$

ii) $S - iS = -100i$

$S(1-i) = -100i$

$S = \dfrac{-100i}{1-i} \times \dfrac{1+i}{1+i}$

$= \dfrac{-100i + 100}{2}$

$= 50 - 50i$

COMMENT:

This was not done particularly well. Students who simplified early struggled to see how $S - iS$ was helpful.

Students could gain full marks in (ii) if they used the wrong answer obtained in (i).

Question 12 (16 marks) Use a SEPARATE writing booklet

(a) The motion of a particle moving along a straight line is given by the equation **2**

$$\frac{d^2x}{dt^2} = -9x.$$

If $x = 0$ and $v = -9$ when $t = 0$, find its displacement at any time t in the form $x = A\sin(nt+\phi)$, where $n, A \in \mathbb{R}^+$ and $\phi \in [0, \pi]$.

The motion described above fits the pattern for SHM where $\ddot{x} = -n^2x$.
Hence, n is equal to 3 and $x = A\sin(3t+\phi)$

Also, $\frac{dx}{dt} = 3A\cos(3t+\phi)$ and using the initial conditions:

$0 = A\sin\phi$ (1) and $-9 = 3A\cos\phi$ (2)

From (1), $\phi = k\pi$ and from the Domain, k can only be 0 or 1 ($A \neq 0$).

Using the results from (1), (2) becomes $-9 = 3A\cos k\pi$

After trying values for k, we get $A = -3$ or 3. However the question states that A must be positive, hence we'll use $A = 3$, which corresponds to $k = 1$.

Finally, $x = 3\sin(3t+\pi)$.

NOTE: Almost every single pupil used integration to find this result.

This was a much longer process than the method described above.

Many had a negative value for A when the question states that A is positive.
Many pupils found a possible value for ϕ as zero but failed to realise this could also be π.

A simple check for solutions would have been to differentiate their answer two times and rarely did pupils catch their own mistakes in this question.
Some used a negative value for n, which is incorrect.

(b) Find (i) **1**

$$\int \sec^3 x \tan x \, dx = \int \sec x \tan x \sec^2 x \, dx$$

$$= \int f'(x)\left(f(x)\right)^2 dx, \qquad \left[f(x) = \sec x\right]$$

$$= \frac{\left(f(x)\right)^3}{3} C, \qquad \left[\text{Reference Sheet}\right]$$

$$= \frac{\sec^3 x}{3} + C$$

(ii) **2**

$$\int \frac{5}{(x+2)(x-3)} dx = \int -\frac{1}{x+2} + \frac{1}{x-3} dx$$

$$= -\ln|x+2| + \ln|x-3| + C$$

$$= \ln\left|\frac{x-3}{x+2}\right| + C$$

NOTE: Pupils could not get full marks if they did not apply absolute value brackets to the arguments of the natural logs (since it is possible for these arguments to be negative, given the domain).

(c) Evaluate the following definite integrals:

(i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x \, dx$. **3**

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \sin x \, dx$

$$I = \left[e^x \sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cos x \, dx$$

$$= e^{\frac{\pi}{2}} \sin\tfrac{\pi}{2} - e^{-\frac{\pi}{2}} \sin\left(-\tfrac{\pi}{2}\right) - \left[e^x \cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \left(-\sin x\right) dx$$

$$= e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}} - e^{\frac{\pi}{2}} \cos\tfrac{\pi}{2} - e^{-\frac{\pi}{2}} \cos\left(-\tfrac{\pi}{2}\right) - I$$

$$2I = e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}$$

$$I = \frac{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{2}$$

NOTE: A common mistake was to not apply the fact that $\sin\left(-\frac{\pi}{2}\right) = -1$ and pupils ended up with $\dfrac{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}{2}$ as an answer.

Some pupils did not even apply the bounds to the primitive of certain parts they found and others mistakenly put $\dfrac{\pi}{2}$ instead of $-\dfrac{\pi}{2}$ for the lower bound in the solution.

(ii) $\displaystyle\int_{-1}^{1} \frac{\sin\frac{2\pi}{5}}{x^2 + 2x\cos\frac{2\pi}{5} + 1}\,dx$ **3**

(Hint: Express the denominator in the form $(x+a)^2 + b^2$)

$$\int_{-1}^{1} \frac{\sin\frac{2\pi}{5}}{x^2 + 2x\cos\frac{2\pi}{5} + 1}\,dx = \int_{-1}^{1} \frac{\sin\frac{2\pi}{5}}{\left(x + \cos\frac{2\pi}{5}\right)^2 + \sin^2\frac{2\pi}{5}}\,dx$$

$$= \left[\tan^{-1}\left(\frac{x + \cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}\right)\right]_{-1}^{1}$$

$$= \tan^{-1}\left(\frac{1 + \cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}\right) - \tan^{-1}\left(\frac{-1 + \cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}\right)$$

$$= \tan^{-1}\left(\frac{1 + 2\cos^2\frac{\pi}{5} - 1}{2\sin\frac{\pi}{5}\cos\frac{\pi}{5}}\right) - \tan^{-1}\left(\frac{-1 + 1 - 2\sin^2\frac{\pi}{5}}{2\sin\frac{\pi}{5}\cos\frac{\pi}{5}}\right)$$

$$= \tan^{-1}\left(\cot\tfrac{\pi}{5}\right) - \tan^{-1}\left(\tan\left(-\tfrac{\pi}{5}\right)\right)$$

$$= \tan^{-1}\left(\tan\left(\tfrac{\pi}{2} - \tfrac{\pi}{5}\right)\right) - \tan^{-1}\left(\tan\left(-\tfrac{\pi}{5}\right)\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{5} - \left(-\frac{\pi}{5}\right)$$

$$= \frac{\pi}{2}$$

NOTE: Many pupils put as their answer, the primitive with the bounds applied and went no further. Thus they could not receive full marks.

(d) Consider the two integrals $I = \int_{-1}^{1} \frac{1}{\left(1+x^2\right)^2} dx$ and $J = -\int_{-1}^{1} \frac{t^2}{\left(1+t^2\right)^2} dt$.

(i) Explain why the use of the substitution $x = \frac{1}{t}$ does not demonstrate that $I = J$. **1**

When x is zero, t is infinite which means this substitution should be done with 2 integrals with an infinite boundary on each. Another valid response is that the substitution $x = \frac{1}{t}$ must be continuous and one to one over the interval of x.

NOTE: Pupils attempted to use the substitution to show these are not correct, but they should've ended up with $I = J$.
The question states it's not correct and wants pupils to explain why.
Many pupils wasted time on this and failed to address the question in any way.
Some pupils gave a vague or generic response and if this were the case, could not receive full marks without giving more details to explain their position.

(d) (ii) Evaluate *I*. **2**

The integrand is an even function, hence $I = 2\int_0^1 \frac{1}{\left(1+x^2\right)^2}\,dx$

Let $x = \tan\theta$, $\frac{dx}{d\theta} = \sec^2\theta$, when $x = 1$, $\theta = \frac{\pi}{4}$ and when $x = 0$, $\theta = 0$.

$$I = 2\int_0^{\frac{\pi}{4}} \frac{1}{\left(1+\tan^2\theta\right)^2}\sec^2\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} + \frac{\sin\frac{\pi}{2}}{2} - \left(0 + \frac{\sin 0}{2}\right)$$

$$= \frac{\pi}{4} + \frac{1}{2}$$

(d) (iii) Evaluate J. **2**

The integrand is an even function, hence $J=-2\int_0^1 \frac{t^2}{\left(1+t^2\right)^2}dt$

Let $x=\tan\theta$, $\frac{dx}{d\theta}=\sec^2\theta$, when $x = 1$, $\theta=\frac{\pi}{4}$ and when $x = 0$, $\theta=0$.

$$I=-2\int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\left(1+\tan^2\theta\right)^2}\sec^2\theta\, d\theta$$

$$=-2\int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\sec^2\theta}d\theta$$

$$=-2\int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta$$

$$=2\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta-1}{2}d\theta$$

$$=\left[\frac{\sin 2\theta}{2}-\theta\right]_0^{\frac{\pi}{4}}$$

$$=\frac{\sin\frac{\pi}{2}}{2}-\frac{\pi}{4}-\left(\frac{\sin 0}{2}-0\right)$$

$$=\frac{1}{2}-\frac{\pi}{4}$$

NOTE: Some pupils tried to do this without a trigonometric substitution, however none were successful in doing so. A few pupils tried $x = \sin\ \theta$, but this gave $1+\sin^2\theta$ in the denominator which doesn't end up being particularly useful. This should have been an indicator for them to try a different substitution, but all who did this pushed through with their substitution and failed to get full marks.

End of Question 12 Solutions

Question (13)

(a) $\ddot{x} = -2e^{-x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2e^{-x}$

$\frac{1}{2}v^2 = 2\int(-e^{-x})\,dx$

$\frac{1}{2}v^2 = 2e^{-x} + c_1$

$x=0, \quad v=2 \implies c_1 = 0$

$\therefore \quad v^2 = 4/e^x$ $\qquad v = \frac{dx}{dt}$

$v = 2e^{-x/2}$

$\frac{dx}{dt} = 2e^{-x/2}$

separating the variables

$\frac{dt}{dx} = \frac{1}{2}e^{x/2}$

$\therefore \quad t = e^{x/2} + c_2$

$t=0, \quad x=0, \implies c_2 = -1$

$e^{x/2} = 1+t, \qquad x/2 = \ln|1+t|$

$\therefore \quad x = 2\ln|1+t|$

(b) $D(x) = x - \tan^{-1}x$

$D'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2}$

NOW $D(0) = 0, \quad D'(0) = 0 \iff x=0$

$D'(x) > 0 \quad \forall x > 0 \ (\because x^2 > 0)$

$\therefore$ $D(x)$ is an increasing function

and $D(x) > 0 \quad \forall x > 0$

Question (13)

$$\frac{d\dot{y}}{dt} = -(g + k\dot{y}^2)$$

$$\therefore \int_{u\sin\alpha}^{\dot{y}} \left(\frac{-d\dot{y}}{g+k\dot{y}^2}\right) = \int_0^t dt$$

$$\therefore \frac{1}{\sqrt{kg}}\left[\tan^{-1}\sqrt{\frac{k}{g}}\,\dot{y}\right]_{\dot{y}}^{u\sin\alpha} = t$$

$$\Longrightarrow t = \frac{1}{\sqrt{kg}}\left[\tan^{-1}\left(\sqrt{\frac{k}{g}}\,u\sin\alpha\right) - \tan^{-1}\sqrt{\frac{k}{g}}\,\dot{y}\right]$$

When particle reaches a maximum height $\dot{y} = 0$, $t = T_1$

$$\therefore T_1 = \frac{1}{\sqrt{kg}}\tan^{-1}\left(\sqrt{\frac{k}{g}}\,u\sin\alpha\right)$$

(iii) $\dot{y} = 0$ is also the condition for maximum height in the absence of air resistance.

$$0 = u\sin\alpha - gT_2 \quad \therefore T_2 = \frac{u\sin\alpha}{g}$$

$$\text{i.e. } T_2 = \frac{1}{\sqrt{kg}}\left(\sqrt{\frac{k}{g}}\,u\sin\alpha\right)$$

$$\therefore T_2 - T_1 = \frac{1}{\sqrt{kg}}\left[\left(\sqrt{\frac{k}{g}}\,u\sin\alpha\right) - \tan^{-1}\left(\sqrt{\frac{k}{g}}\,u\sin\alpha\right)\right]$$

Let $X = \sqrt{\frac{k}{g}}\,u\sin\alpha$

$$\therefore T_2 - T_1 = \frac{1}{\sqrt{kg}}(X - \tan^{-1}X), \quad \forall X > 0$$

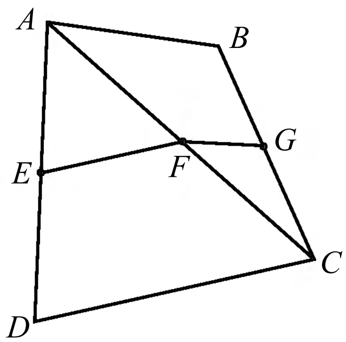
$$> 0 \quad [\text{from (b) (i)}]$$

$$\Longrightarrow T_2 - T_1 > 0 \quad \therefore T_2 > T_1$$

Question 13 (continued)

(c) $ABCD$ is a quadrilateral. **3**

E, F and G are the midpoints of AD, AC, and BC respectively.



Using vector methods, show that $\overrightarrow{AB} + \overrightarrow{DC} = 2\,\overrightarrow{EG}$.

$\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AD} \quad [E \text{ is midpoint of } AD]$

$\overrightarrow{AF} = \frac{1}{2}\overrightarrow{AC} \quad [F \text{ is midpoint of } AC]$

$\overrightarrow{BG} = \frac{1}{2}\overrightarrow{BC} \quad [G \text{ is midpoint of } BC]$

$\overrightarrow{EG} = \overrightarrow{EF} + \overrightarrow{FG}$

$= \left(\overrightarrow{AF} - \overrightarrow{AE}\right) + \left(\overrightarrow{FC} - \overrightarrow{GC}\right)$

$= \left(\frac{1}{2}\overrightarrow{AC} - \frac{1}{2}\overrightarrow{AD}\right) + \left(\frac{1}{2}\overrightarrow{AC} - \frac{1}{2}\overrightarrow{BC}\right)$

$= \frac{1}{2}\left(\overrightarrow{AC} - \overrightarrow{AD}\right) + \frac{1}{2}\left(\overrightarrow{AC} - \overrightarrow{BC}\right)$

$= \frac{1}{2}\overrightarrow{DC} + \frac{1}{2}\overrightarrow{AB}$

$\therefore \overrightarrow{EG} = \frac{1}{2}\overrightarrow{DC} + \frac{1}{2}\overrightarrow{AB}$

$\therefore 2\overrightarrow{EG} = \overrightarrow{DC} + \overrightarrow{AB}$

Comments on Question 13

(a) Generally well done although a considerable number of students starts off with $v\frac{dv}{dx}$ and then integrating with respect to t. Students are the most successful when they use $\frac{d}{dx}(\frac{1}{2}v^2) = -2e^{-x}$

Common error was the inability to evaluate the constants or integration through initial conditions

(b)

(i) Most can differentiate properly to derive $D'(x) = 1 - \dfrac{1}{1+x^2}$ but cannot show that $D'(x) = \dfrac{x^2}{1+x^2}$ and argue that $D'(x) > 0 \qquad (x^2 > 0,\ 1 + x^2 > 0)$

(ii)

- Most use v as the $\dot{y}$ in the differential equation $\frac{d\dot{y}}{dt} = -(g + k\dot{y}^2)$ and then proceeded to integrate with respect to v
- Almost all the students recognise that the particle reaches a maximum height then $\dot{y} = 0$
- Quite a few students went straight to

  $$\int_{u\sin\alpha}^{0} \frac{-d\dot{y}}{y + k\dot{y}^2}$$

  and derive

  $$T_1 = \frac{1}{\sqrt{kg}} \tan^{-1} \sqrt{\frac{k}{g}} u \sin\alpha$$

(iii) Although most students aware that they should by (b)(i),
most just pluck in $u\sin\alpha$ as x and then $\left(\sqrt{\frac{k}{y}}\, u\sin\alpha\right)$ as the x in $\tan^{-1}\sqrt{\frac{k}{g}} u\sin\alpha$ and try to fudge the answers.

(c) This section was very hard and only a handful were successful in manipulating vectors
e.g. most could not recognise

$$\begin{aligned} 2\vec{EG} &= 2\vec{AB} + \vec{BC} - \vec{AD} \\ &= \vec{AB} + \underbrace{(\vec{BC} - \vec{BA})}_{\vec{AC}} - \vec{AD} \qquad \text{(break } \vec{AB} \text{ into 2 parts)} \end{aligned}$$

Question 14

(a) Let $S(n)$ be the proposition that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}$$

$n \in \mathbb{Z}^+,\ n>1$

For $n>1$, (say $n=2$)

$$1+\frac{1}{\sqrt{2}}-\sqrt{2}$$

$$=\frac{\sqrt{2}+1-2}{\sqrt{2}}=\frac{\sqrt{2}-\sqrt{1}}{\sqrt{2}}>0$$

i.e $1+\frac{1}{\sqrt{2}}>\sqrt{2}$. $\therefore S(2)$ is true.

Assume $S(k)$ is true. ie

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}>\sqrt{k},\quad k\in\mathbb{Z}^+$$

Consider $S(k+1)$

$$\underbrace{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}}+\frac{1}{\sqrt{k+1}}$$

$$>\sqrt{k}+\frac{1}{\sqrt{k+1}}=\frac{\sqrt{k}(\sqrt{k+1})+1}{\sqrt{k+1}}$$

(From inductive hypothesis)

$$=\frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}}$$

$$>\frac{k+1}{\sqrt{k+1}}\quad \left(\because \sqrt{k(k+1)}>\sqrt{k^2} \text{ when } k>\right)$$

$$=\sqrt{k+1}$$

$\Rightarrow S(k+1)$ is also true.

Hence by the principle of Mathematical Induction

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n},\quad n\in\mathbb{Z}^+,\ n>1$$

(b) $$\int \frac{\cos x \, dx}{\sin x \sqrt{1+\sin x}}$$

Let $u^2 = 1 + \sin x \Rightarrow \sin x = u^2 - 1$

$$2u \, du = \cos x \, dx$$

$$\therefore \int \frac{\cos x \, dx}{\sin x \sqrt{1+\sin x}} = 2 \int \frac{\cancel{u} \, du}{(u^2-1)\cancel{u}}$$

$$= 2 \int \frac{du}{u^2-1}$$

$$\frac{2}{u^2-1} = 2\left(\frac{a}{u+1} + \frac{b}{u-1}\right)$$

$$a+b=0 \Rightarrow b=-a, \quad -2a=1$$

$$\therefore a = -\tfrac{1}{2}, \quad b = \tfrac{1}{2}.$$

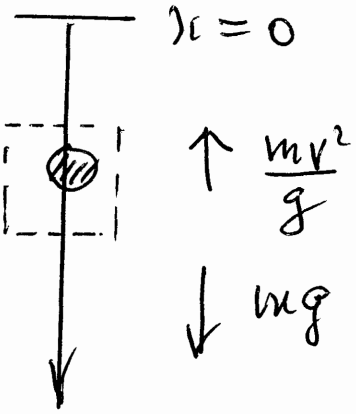
$$\therefore \frac{2}{u^2-1} = \frac{1}{u-1} - \frac{1}{u+1}$$

$$\therefore 2\int \frac{du}{u^2-1} = \ln\left(\frac{u-1}{u+1}\right)$$

$$= \ln\left[\frac{\sqrt{1+\sin x} - 1}{\sqrt{1+\sin x} + 1}\right]$$

(c)

(i)



$$\cancel{m}\ddot{x} = \cancel{m}g - \frac{\cancel{m}v^2}{g}$$

$$\therefore \ddot{x} = g - \frac{v^2}{g}$$

$$\therefore \ddot{x} = \frac{g^2 - v^2}{g}$$

(ii) Separating the variables

$$\frac{dt}{dv} = \frac{g}{g^2 - v^2}$$

$$\therefore t = g\int \frac{dv}{(g+v)(g-v)}$$

$\boxed{\begin{array}{l} \text{partial fractions} \\ 1 = a(g-v) + b(g+v) \\ -a+b=0 \Rightarrow a=b,\quad 1 = 2ag \\ \therefore a = \frac{1}{2g} = b. \end{array}}$

$$\frac{g}{g^2-v^2} = \left(\frac{a}{g+v} + \frac{b}{g-v}\right)g$$

$$t = \int \frac{dv}{g^2-v^2} = \frac{1}{2}\int\left(\frac{1}{g+v} - \frac{1}{g-v}\right)dv$$

$$\therefore t = \frac{1}{2}\ln\left(\frac{g+v}{g-v}\right)$$

i.e. $2t = \ln\left(\frac{g+v}{g-v}\right)$

$$e^{2t} = \frac{g+v}{g-v}$$

$$ge^{2t} - ve^{2t} = g+v$$

$$\therefore v = g\left(\frac{e^{2t}-1}{e^{2t}+1}\right)$$

Also $\ddot{x} = v\frac{dv}{dx}.$

$$v\frac{dv}{dx} = \frac{g^2-v^2}{g}.$$

$$\frac{dx}{dv} = \frac{gv}{g^2-v^2}$$

Separating the variables:

$$\int dx = -\frac{g}{2}\int \frac{(-2v)\,dv}{g^2 - v^2}$$

$$\therefore \quad x = -\frac{g}{2}\ln(g^2 - v^2) + C_1$$

When $x = 0$, $v = 0 \Rightarrow C_1 = \frac{g}{2}\ln(g^2)$

$$\therefore \quad x = \frac{g}{2}\left[\ln\left(\frac{g^2}{g^2 - v^2}\right)\right]$$

$$e^{\frac{2x}{g}} = \frac{g^2}{g^2 - v^2}.$$

$$e^{-\frac{2x}{g}} = \frac{g^2 - v^2}{g^2}$$

$$\therefore \quad g^2 - v^2 = g^2 e^{-\frac{2x}{g}}$$

$$\therefore \quad v^2 = g^2\left(1 - e^{-\frac{2x}{g}}\right).$$

(iii) When $\ddot{x} \to 0$, $v \to v_T \to g$

Where v_T is the terminal velocity.

$$\therefore \quad v_T = g \Rightarrow v_T/2 = g/2.$$

$$\frac{g}{2} = g\left(\frac{e^{2t} - 1}{e^{2t} + 1}\right)$$

$$\therefore \quad 2(e^{2t} - 1) = e^{2t} + 1, \quad t = \frac{\ln 3}{2}$$

$$v_T = g, \quad v^2 = g^2\left(1 - e^{-2x/g}\right)$$

$$\frac{g^2}{4} = g^2\left(1 - e^{2x/g}\right)$$

$$e^{-\frac{2x}{g}} = 3/4, \quad \frac{2x}{g} = \ln(4/3)$$

$$\therefore \quad x = \frac{g}{2}\ln(4/3)$$

Comments on Question 14

(a) This induction question is generally not done well.

- Many checked the case $n = 1$ where as the question specified $n > 1$
- Those students went on to prove S(2) never did quite that merely stating $1 + \dfrac{1}{\sqrt{2}} > \sqrt{2}$
- I like to think of inequalities (in applying the inductive hypothesis) as "the art of tossing out the garbage" e.g.

$$\begin{aligned}\sqrt{k} + \frac{1}{\sqrt{k+1}} &= \frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}} \\ &> \frac{k+1}{\sqrt{k+1}} \qquad (\sqrt{k(k+1)} > \sqrt{k^2})\end{aligned}$$

  Only a handful took this path, most just bogged down with the algebra

(b) This was generally well done

- The problems came from an inability to handle surdic expressions as a result of the substitution $u = 1 + \sin x$
- Substitution students believed that $\dfrac{1}{u^{\frac{3}{2}} - u} = \dfrac{1}{u^{\frac{3}{2}}} - \dfrac{1}{u}$
- Using 't' formulae has the worse scenario in this integration.

(c)

(i) Well done. Most students know how to divide m and then express the expression as a fraction.

(ii) The integration and separating step was moderately well done even though most had no problem with converting the expression as a partial fraction quite a few students use $v\frac{dv}{dx}$ and the carry on integrating with respect to t.

(iii) Carelessness in solving equation $2(e^{2t} - 1) = e^{2t} + 1$ ends up somehow $t = \frac{1}{2}\ln 2$ instead of $t = \frac{1}{2}\ln 3$

Question 15 (16 marks)

(a) Let $t = \tan\dfrac{x}{2}$.

(i) Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+k\sin x}\,dx = \frac{2}{\sqrt{1-k^2}}\tan^{-1}\left(\sqrt{\frac{1-k}{1+k}}\right)$, where $0 < k < 1$. **4**

$\dfrac{dt}{dx} = \dfrac{\sec^2\frac{x}{2}}{2}$, hence $dx = 2\cos^2\frac{x}{2}\,dt$, where $\cos\frac{x}{2} = \dfrac{1}{\sqrt{1+t^2}}$

When $x = \dfrac{\pi}{2}$, $t = 1$ and when $x = 0$, $t = 0$.

The integral becomes:

$$\int_0^1 \frac{2}{\left(1+\frac{2kt}{1+t^2}\right)}\frac{dt}{\left(1+t^2\right)} = \int_0^1 \frac{2}{1+t^2+2kt}\,dt$$

$$= 2\int_0^1 \frac{1}{(t+k)^2+\left(\sqrt{1-k^2}\right)^2}\,dt$$

$$= \frac{2}{\sqrt{1-k^2}}\left[\tan^{-1}\left(\frac{t+k}{\sqrt{1-k^2}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{1-k^2}}\left(\tan^{-1}\left(\frac{1+k}{\sqrt{1-k^2}}\right) - \tan^{-1}\left(\frac{k}{\sqrt{1-k^2}}\right)\right)$$

$$= \frac{2}{\sqrt{1-k^2}}\tan^{-1}\left(\frac{\frac{1+k}{\sqrt{1-k^2}} - \frac{k}{\sqrt{1-k^2}}}{1+\left(\frac{1+k}{\sqrt{1-k^2}}\right)\left(\frac{k}{\sqrt{1-k^2}}\right)}\right)$$

$$= \frac{2}{\sqrt{1-k^2}}\tan^{-1}\left(\frac{\frac{1}{\sqrt{1-k^2}}}{1+\frac{k+k^2}{1-k^2}}\right)$$

$$= \frac{2}{\sqrt{1-k^2}}\tan^{-1}\left(\frac{\frac{1-k^2}{\sqrt{1-k^2}}}{1-k^2+k+k^2}\right)$$

$$= \frac{2}{\sqrt{1-k^2}}\tan^{-1}\left(\frac{\sqrt{1-k^2}}{1+k}\right)$$

$$= \frac{2}{\sqrt{1-k^2}}\tan^{-1}\left(\frac{\sqrt{1-k}\sqrt{1+k}}{1+k}\right)$$

$$= \frac{2}{\sqrt{1-k^2}}\tan^{-1}\sqrt{\frac{1-k}{1+k}}$$

NOTE: For full marks, a worked solution for finding this single expression after applying the boundaries of the integral to the primitive function of the integrand was required.

(a) Let $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{2 + \sin x}\, dx$, where $n = 0, 1, 2, \ldots$.

(ii) Show that $I_{n+1} + 2I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$. **1**

$$I_{n+1} = \int_0^{\frac{\pi}{2}} \frac{\sin^{n+1} x}{2 + \sin x}\, dx$$

$$\begin{aligned} I_{n+1} + 2I_n &= \int_0^{\frac{\pi}{2}} \frac{\sin^{n+1} x + 2\sin^n x}{2 + \sin x}\, dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin^n x(\sin x + 2)}{2 + \sin x}\, dx \\ &= \int_0^{\frac{\pi}{2}} \sin^n x\, dx \end{aligned}$$

NOTE: This was well done by most pupils.

(a) (iii) Using parts (i) and (ii), find the value of I_2. **3**

Give your answer in the form $m\pi+1$, where m is irrational in simplest form.

$$I_2+I_1=\int_0^{\frac{\pi}{2}}\sin x\,dx\text{, then}$$

$$I_2=\left[-\cos x\right]_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\frac{\sin x}{2+\sin x}\,dx$$

$$=-\cos\tfrac{\pi}{2}--\cos 0-2\int_0^{\frac{\pi}{2}}\left(\frac{\sin x+2}{2+\sin x}-\frac{2}{2+\sin x}\right)dx$$

$$=1+2\int_0^{\frac{\pi}{2}}\left(\frac{2}{2+\sin x}-1\right)dx$$

$$=1-2\left[x\right]_0^{\frac{\pi}{2}}+2\int_0^{\frac{\pi}{2}}\frac{1}{1+\frac{\sin x}{2}}\,dx$$

$$=1-2\left(\frac{\pi}{2}-0\right)+\frac{4}{\sqrt{1-\left(\frac{1}{2}\right)^2}}\tan^{-1}\left(\sqrt{\frac{1-\frac{1}{2}}{1+\frac{1}{2}}}\right)$$

$$=1-\pi+\frac{8}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$$

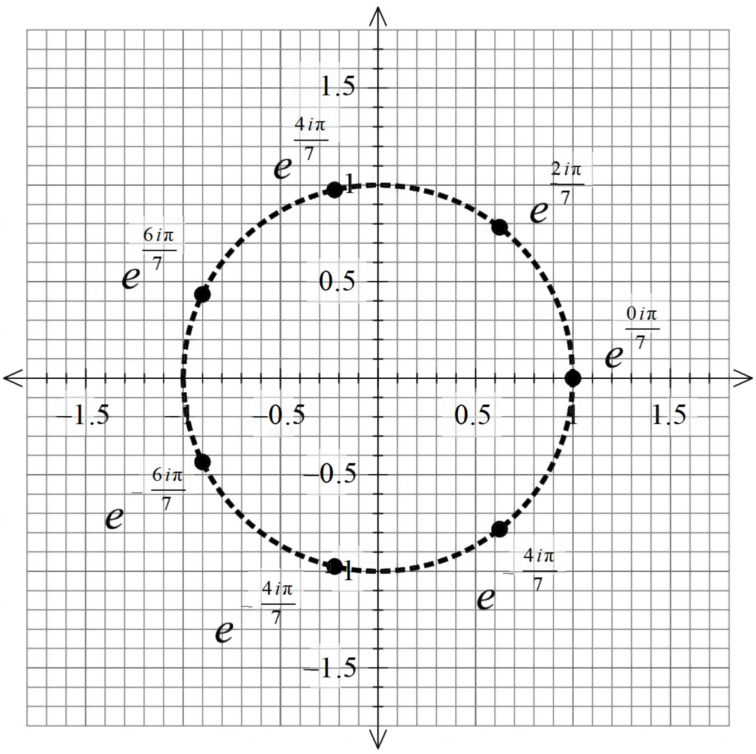
$$=1-\pi+\frac{8\pi}{6\sqrt{3}}$$

$$=1+\frac{\pi}{9}\left(4\sqrt{3}-9\right)$$

NOTE: Some pupils used an expression like the first line in the solution, except they had $\sin^2 x$ as the integrand, which is incorrect.

(b) Let $P(z) = z^7 - 1$.

(i) Write your solutions to $P(z) = 0$ on an Argand diagram. **1**



NOTE: Marks were awarded for the diagram pupils made and not for their working for reasons that should not require explanation.
Pupils needed to state what the points were on their diagram or at least show the moduli of each value and arguments between values.
There needed to be something explicit about their points to show their values.

(b) (ii) Show that $P(z)=z^3(z-1)\left[\left(z+\frac{1}{z}\right)^3+\left(z+\frac{1}{z}\right)^2-2\left(z+\frac{1}{z}\right)-1\right]$ **3**

$z+\frac{1}{z}=z+z^{-1}$, then

$$\left(z+z^{-1}\right)^3=z^3+3z+3z^{-1}+z^{-3} \text{ and } \left(z+z^{-1}\right)^2=z^2+2+z^{-2}$$

$$\begin{aligned} P(z) &= z^3(z-1)\left[\left(z^3+3z+3z^{-1}+z^{-3}\right)+\left(z^2+2+z^{-2}\right)-2z-2z^{-1}-1\right] \\ &= (z-1)\left[\left(z^6+3z^4+3z^2+1\right)+\left(z^5+2z^3+z\right)-2z^4-2z^2-z^3\right] \\ &= (z-1)\left(z^6+z^5+z^4+z^3+z^2+z+1\right) \\ &= z^7-1 \end{aligned}$$

NOTE: Many pupils did this 'backwards' starting with the expression for $P(z)$ in part (i). Pupils could not get full marks if they did not enough detail.

(iii) Hence, solve the equation $x^3+x^2-2x-1=0$ **2**

From part (ii), some of the solutions of $P(z)$ are solutions to this equation.
The modulus of all the solutions of $P(z)$ is one, hence: $\frac{1}{z}=\overline{z}$ and the equation above can be rewritten as a real function in the variable z + $\overline{z}$.

$$f(x)=x^3+x^2-2x-1$$
$$f\left(z+\overline{z}\right)=\left(z+\overline{z}\right)^3+\left(z+\overline{z}\right)^2-2\left(z+\overline{z}\right)-1$$

Which is the last factor in function in part (ii).

The solutions are the sum of conjugate pairs of solutions to $P(z)$.

Solutions are $2\cos\left(\frac{2\pi}{7}\right)$, $2\cos\left(\frac{4\pi}{7}\right)$ and $2\cos\left(\frac{6\pi}{7}\right)$.

NOTE: Some pupils gave this answer in sine form but often failed to realise that this would mean that two of the solutions here would then have a negative sign in front.

(b) (iv) Hence prove that $\operatorname{cosec}\dfrac{\pi}{14}\operatorname{cosec}\dfrac{3\pi}{14}\operatorname{cosec}\dfrac{5\pi}{14}=8.$ **2**

From part (iii), the product of the roots is 1.

$$2\cos\left(\tfrac{2\pi}{7}\right)\times 2\cos\left(\tfrac{4\pi}{2}\right)\times 2\cos\left(\tfrac{6\pi}{2}\right)=1$$

$$\begin{aligned}
8 &= \sec\left(\tfrac{2\pi}{7}\right)\sec\left(\tfrac{4\pi}{7}\right)\sec\left(\tfrac{6\pi}{7}\right)\\
8 &= \cos ec\left(\tfrac{\pi}{2}-\tfrac{2\pi}{7}\right)\cos ec\left(\tfrac{\pi}{2}-\tfrac{4\pi}{7}\right)\cos ec\left(\tfrac{\pi}{2}-\tfrac{6\pi}{7}\right)\\
8 &= \cos ec\left(\tfrac{3\pi}{14}\right)\cos ec\left(-\tfrac{\pi}{14}\right)\cos ec\left(-\tfrac{5\pi}{14}\right)\\
8 &= \cos ec\left(\tfrac{\pi}{14}\right)\cos ec\left(\tfrac{3\pi}{14}\right)\cos ec\left(\tfrac{5\pi}{14}\right)
\end{aligned}$$

NOTE: A number of pupils started with the result and needed to state that their solution relies on the product of roots of the equation in part (iii).

Question 16

a) i) $1 - x^2 + x^4 - \dots + (-1)^n x^{2n}$ $\qquad S_n = \dfrac{a(1-r^n)}{1-r}$

$$= \frac{1\left(1-(-x^2)^{n+1}\right)}{1-(-x^2)}$$

$$= \frac{1-(-1)^{n+1}.x^{2n+2}}{1+x^2}$$

$$= \frac{1+(-1)^n.x^{2n+2}}{1+x^2}$$

ii) $$\int_0^1 \left(1 - x^2 + x^4 - \dots + (-1)^n x^{2n}\right) dx = \int_0^1 \frac{1+(-1)^n.x^{2n+2}}{1+x^2}\,dx$$

$$\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n.\frac{x^{2n+1}}{2n+1}\right]_0^1 = \int_0^1 \left(\frac{1}{1+x^2} + \frac{(-1)^n.x^{2n+2}}{1+x^2}\right) dx$$

$$(1) - \frac{(1)^3}{3} + \frac{(1)^5}{5} - \dots + (-1)^n.\frac{(1)^{2n+1}}{2n+1} - (0) = \left[\tan^{-1}x\right]_0^1 + (-1)^n \int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx$$

$$1 - \frac{1}{3} + \frac{1}{5} - \dots + (-1)^n.\frac{1}{2n+1} = \tan^{-1}1 - \cancel{\tan^{-1}0} + (-1)^n \int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx$$

$$S_n = \frac{\pi}{4} + (-1)^n \int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx$$

$\therefore A = \dfrac{\pi}{4}$

iii) $$S_n - A = (-1)^n \int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx$$

$$|S_n - A| = \left|(-1)^n \int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx\right|$$

$$= \int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx$$

For $0 < x < 1$

$0 < x^2 < 1$

$1 < 1 + x^2 < 2$

$\frac{1}{1} > \frac{1}{1+x^2} > \frac{1}{2}$

$\frac{1}{2} < \boxed{\frac{1}{1+x^2} < 1}$

$\therefore \quad \frac{x^{2n+2}}{1+x^2} < x^{2n+2}$

And so $\int_0^1 \frac{x^{2n+2}}{1+x^2} < \int_0^1 x^{2n+2}\, dx$

$= \left[\frac{x^{2n+3}}{2n+3}\right]_0^1$

$= \frac{(1)^{2n+3}}{2n+3} - (0)$

$= \frac{1}{2n+3}$

$\therefore \quad |S_n - A| < \frac{1}{2n+3}$

iv) $|S_n - A| < \frac{1}{2n+3}$

$-\frac{1}{2n+3} < S_n - A < \frac{1}{2n+3}$

as $n \to \infty$

$\frac{1}{2n+3} \to 0$

$\therefore \quad S_\infty - A = 0$

$S_\infty = A$

ie $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$.

COMMENT:

Very few students made the connection between $1 - x^2 + x^4 - \dots + (-1)^n x^{2n}$ and $S_n = 1 - \frac{1}{3} + \frac{1}{5} - \dots + (-1)^n \frac{1}{2n+1}$

which meant very few students could answer (ii) correctly.

A lot of students had the wrong number of terms for the geometric series in (i)

$1 - x^2 + x^4 - \dots + (-1)^n x^{2n}$

$= 1 - x^{2(1)} + x^{2(2)} - x^{2(3)} + x^{2(4)} - \dots + (-1)^n x^{2(n)}$

Note: counting from 1 to n there are n terms but we have an extra term.

Even without gaining the correct answer in (ii) Parts (iii) & (iv) could and should have been attempted.

In general, not many marks were awarded in this question.

b) i) $a_1 a_2 - \left(\frac{a_1 + a_2}{2}\right)^2$

$= a_1 a_2 - \left(\frac{a_1^2 + 2a_1a_2 + a_2^2}{4}\right)$

$= \frac{4a_1a_2 - a_1^2 - 2a_1a_2 - a_2^2}{4}$

$= -\frac{a_1^2 - 2a_1a_2 + a_2^2}{4}$

$= -\frac{(a_1 - a_2)^2}{4}$

≤ 0

$\therefore\ a_1 a_2 \le \left(\dfrac{a_1 + a_2}{2}\right)^2$

ii) Prove true for $m=1$

$LHS = a_1 \times a_2$

$RHS = \left(\dfrac{a_1 + a_2}{2^1}\right)^{2^1}$

$LHS \le RHS$ using (i)

$\therefore$ true for $m=1$

Assume true for $m=k$, $k \in \mathbb{Z}^+$

$$a_1 \times a_2 \times a_3 \times \dots \times a_{2^k-1} \times a_{2^k} \le \left(\frac{a_1 + a_2 + a_3 + \dots + a_{2^k-1} + a_{2^k}}{2^k}\right)^{2^k}$$

Prove true for $m=k+1$

ie $$a_1 \times a_2 \times a_3 \times \dots \times a_{2^{k+1}-1} \times a_{2^{k+1}} \le \left(\frac{a_1 + a_2 + a_3 + \dots + a_{2^{k+1}-1} + a_{2^{k+1}}}{2^{k+1}}\right)^{2^{k+1}}$$

$$LHS = a_1 \times a_2 \times a_3 \times \dots \times a_{2^k} \times a_{2^k+1} \times a_{2^k+2} \times a_{2^k+3} \times \dots \times a_{2^k+2^k}$$

$$\le \left(\frac{a_1 + a_2 + a_3 + \dots + a_{2^k}}{2^k}\right)^{2^k} \times a_{2^k+1} \times a_{2^k+2} \times a_{2^k+3} \times \dots \times a_{2^k+2^k} \quad \text{using assumption}$$

$$\le \left(\frac{a_1 + a_2 + a_3 + \dots + a_{2^k}}{2^k}\right)^{2^k} \left(\frac{a_{2^k+1} + a_{2^k+2} + a_{2^k+3} + \dots + a_{2^{k+1}}}{2^k}\right)^{2^k} \quad \text{using assumption again.}$$

$$= \left[\left(\frac{a_1 + a_2 + a_3 + \dots + a_{2^k}}{2^k}\right)\left(\frac{a_{2^k+1} + a_{2^k+2} + a_{2^k+3} + \dots + a_{2^{k+1}}}{2^k}\right)\right]^{2^k}$$

$$\le \left[\left[\frac{\dfrac{a_1 + a_2 + a_3 + \dots + a_{2^k}}{2^k} + \dfrac{a_{2^k+1} + a_{2^k+2} + a_{2^k+3} + \dots + a_{2^{k+1}}}{2^k}}{2}\right]^2\right]^{2^k} \quad \text{using (i)}$$

$$= \left[\frac{a_1 + a_2 + a_3 + \dots + a_{2^{k+1}}}{2^{k+1}}\right]^{2^{k+1}}$$

$$= RHS$$

$\therefore$ true for $m = k+1$

$\therefore$ true by induction for all positive integers m.

iii) $$b_1 \times b_2 \times b_3 \times \dots \times b_n \times b_{n+1} \times b_{n+2} \times \dots \times b_{2^m} \leq \left(\frac{b_1 + b_2 + b_3 + \dots + b_n + b_{n+1} + \dots + b_{2^m}}{2^m}\right)^{2^m}$$

$$a_1 \times a_2 \times a_3 \times \dots \times a_n \times \underbrace{A \times A \times \dots \times A}_{2^m - n \text{ terms}} \leq \left(\frac{a_1 + a_2 + a_3 + \dots + a_n + \overbrace{A + A + \dots + A}^{2^m - n \text{ terms}}}{2^m}\right)^{2^m}$$

$$a_1 \times a_2 \times a_3 \times \dots \times a_n \times A^{2m-n} \leq \left(\frac{a_1 + a_2 + a_3 + \dots + a_n + (2^m - n)A}{2^m}\right)^{2^m}$$

$$= \left(\frac{n \cdot \frac{1}{n}(a_1 + a_2 + a_3 + \dots + a_n) + (2^m - n)A}{2^m}\right)^{2^m}$$

$$= \left(\frac{\cancel{nA} + \cancel{2^m} A - \cancel{nA}}{\cancel{2^m}}\right)^{2^m}$$

$$= A^{2m}$$

$$\therefore a_1 \times a_2 \times a_3 \times \dots \times a_n A^{2m-n} \leq A^{2m}$$

iv) $$a_1 \times a_2 \times a_3 \times \dots \times a_n \leq \frac{A^{2m}}{A^{2m-n}}$$

$$a_1 \times a_2 \times a_3 \times \dots \times a_n \leq A^n$$

$$\sqrt[n]{a_1 \times a_2 \times a_3 \times \dots \times a_n} \leq A$$

$$\sqrt[n]{a_1 \times a_2 \times a_3 \times \dots \times a_n} \leq \frac{a_1 + a_2 + a_3 + \dots + a_n}{n}$$

$$\therefore \frac{a_1 + a_2 + a_3 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \times a_2 \times a_3 \times \dots \times a_n}$$

COMMENT:

Part (i) was by considering. There was no otherwise.

Only a couple of students gained full marks in part (ii)
The majority gained two marks or less.

The 2^m that was involved in the induction meant

- there were two terms for $m=1$.
- there were double the number of terms for $m=k+1$ compared with $m=k$.

The added complexity (introducing another pronumeral) confused most students who could not make logical progress towards the result.

Part (iv) should have been an easy mark.

A tough Q16 overall!